NASA TECHNICAL TRANSLATION

NASA TT F-11, 827

NASA TT F-11, 827

A PRELIMINARY STUDY OF ORBITAL CORRECTIONS BY MEANS OF A SMALL THRUST

E. Cambi

Translation of "Preliminares au Probleme des Corrections Orbitales a Faible Poussee."
European Space Vehicle Launcher Development Organization, Paris, August 1967, 35 pp. ELDO-TM-101.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 AUGUST 1968

NASA TT F-11, 827

A PRELIMINARY STUDY OF ORBITAL CORRECTIONS BY MEANS OF A SMALL THRUST

Prof. E. Cambi
General Management of Research
and Experiment

ABSTRACT: A synthesis is presented of methods of orbital transfer by means of a small thrust, especially the "direct" equations expressing the coordinates of the mobile point as a function of time, and the "planetary" equations which express the law of variation of the parameters of an osculating orbit. It can be seen that the latter equations are much more amenable to treatment, especially if one takes the time average of the terms in order to determine the "secular" variations. However, by means of examples it is shown that the latter procedure is dangerous, and that it is in any case preferable to utilize the direct equations.

I. CORRECTIONS IN THE PLANE

1. It is possible to decompose the acceleration of manoeuver (thrust divided by mass) either with respect to a terrestrial reference (R — positive radial acceleration in a centrifugal direction, C — positive circumferential acceleration in the direction of motion), or with respect to an orbital reference (T — tangential acceleration, N — acceleration normal to the orbit and positive in the outer direction). /1*

Denoting by α the angle between the tangent to the orbit and the horizontal, we obtain the simple formulas

$$\begin{array}{lcl} T = C\cos\alpha + R\sin\alpha & ; & C = T\cos\alpha - N\sin\alpha \\ N = -C\sin\alpha + R\cos\alpha & & R = T\sin\alpha + N\cos\alpha \end{array} \qquad (1)$$

By describing the motion in terms of osculating orbits (the osculating orbit during a time t being the ballistic orbit the vehicle would follow if the thrust would cease), we obtain the velocity components

$$\frac{dr}{dt} = \frac{pe\sin\theta}{(1+e\cos\theta)^2}\frac{d\theta}{dt}, \quad \frac{rd\varphi}{dt} = \frac{p}{1+e\cos\theta}\frac{d\theta}{dt}$$

*Numbers in the margin indicate pagination in the foreign text.

in terms of the semilatus rectum p and the eccentricity e, with θ being the true anomaly, r the radius vector, and φ the azimuth with respect to a fixed direction. Since $\frac{d\theta}{dt} = \sqrt{\frac{\mu}{p^3}}(1+e\cos\theta)^2$, we obtain for the absolute value of the velocity

$$v^2 = \frac{\mu_0}{p}(1+e^2+2e\cos\theta) = \frac{\mu_0}{a}\,\frac{1+e\cos E}{1-e\cos E}.$$

where we introduced the eccentric anomaly E. Hence

$$\sin\alpha = \sqrt{\frac{\mu_0}{p}}\;\frac{e\sin\theta}{v} = \frac{e\sin E}{\sqrt{1-e^2\cos^2 E}}$$

$$\cos\alpha = \sqrt{\frac{\mu_0}{p}}\;\frac{1+e\cos\theta}{v} = \frac{\sqrt{1-e^2}}{\sqrt{1-e^2\cos^2 E}}. \qquad (2)$$ /2

The problem can be treated from two different viewpoints: In direct form, by writing down the differential equations of the motion; in "planetary" form, by regarding the trajectory as an envelope of osculating orbits and writing down the differential equations governing the variation of the orbital parameters. The second approach yields more readily integrable equations, in any case if one agrees (by assuming the parameter variations to be slow) to replace the fast varying terms, containing θ or E, by their mean values during one revolution (which causes, however, certain difficulties of interpretation).

By a simple reasoning we can arrive at an immediate tentative conclusion. The vehicle energy per unit mass is

$$\frac{1}{2}v^2 - \frac{\mu_0}{r}$$

This is true only if $-\frac{\mu_0}{2a}$ is the semi major axis of the osculating orbit. On the other hand the derivative of the energy is evidently vT if T is the tangential component of the acceleration; if the object of the manoeuver is an enlargement of the orbit, the only useful component of the thrust will be tangential component and we precisely obtain

$$\frac{d}{dt}\left(-\frac{\mu_0}{2a}\right) = \frac{\mu_0}{2a^2}\,\frac{da}{dt} = vT$$

whence

$$\frac{1}{a_0} - \frac{1}{a} = \frac{2}{\mu_0}\int_0^t vT\,dt \qquad (3)$$

It would be premature, however, to conclude that in order to optimize an enlargement of the orbit the acceleration of manoeuver must be constantly tangential. This condition maximizes the rate of variation of the total energy; but it might be more convenient to maximize at first only the kinetic energy, in order to increase the velocity v as much as possible and to approach the direction of the thrust to the tangent when the velocity has sufficiently increased. If the angle between the tangent and the horizontal is appreciable, the velocity will increase more rapidly when the thrust is nearer to the horizontal than to the tangent. This optimization of the direction program will not be discussed in our "Preliminary Study", which is confined to an examination of the case most directly "amenable" to an analysis which is as complete as possible; this all the more so, in view of the fact that in the case of "small thrusts" the orbit does not differ much at the beginning from the horizontal, while at the end of the operation the thrust must be tangential in any case. The optimization problem was discussed by Lawden (Astronautica Acta, vol. 4, No. 3, p. 218, 1958). /3

In the present paper we confine ourselves to the case of constant acceleration components, which is not compatible with a "Program".

2. Direct formulations

The equations of motion in the radial (R) and circumferential (C) accelerations are

$$\frac{d^2 r}{dt^2} = R + r\left(\frac{d\varphi}{dt}\right)^2 - g_o \frac{r_o^2}{r^2}$$

$$\frac{d}{dt}\left[r^2 \frac{d\varphi}{dt}\right] = r\,C \tag{4}$$

where φ is the geocentric azimuth with respect to the original vertical, and g_o is the acceleration of gravity at an altitude r_o corresponding to the instant of application of the thrust, which yields $\mu_o = g_o r_o^2$.

When the thrust is decomposed into its tangential and normal components, the original equations assume the form

$$\frac{dv}{dt} = T - \frac{\mu_o}{r^2}\sin\alpha$$

$$\frac{v^2}{\rho} = \frac{\mu_o}{r^2}\cos\alpha - N \tag{5}$$

where α is the angle between the orbit and the horizontal, v the instantaneous velocity, and ρ the radius of curvature.

System (4), just as system (5), can be reduced to a single equation with one unknown, provided that the following conditions are satisfied: The components of the acceleration must be constant themselves (which implies the constancy of the absolute value and of the direction of the acceleration with respect to the vertical in case (4), and with respect to the orbit in case (5); but in the latter case it is also necessary that N = O, which is not a major restriction, since we know that the N component does not yield any useful work for enlarging the orbit). /4

Such a reduction to a single equation can be of interest only for the purpose of a manual solution. But this is not feasible, since the only resultant equation is generally nonlinear and intractable. The only case in which the solution can be reduced to a single quadrature is the case R = const (a purely vertical constant thrust), which is not of great interest in view of its low efficiency, i.e., a radial thrust is in fact almost perpendicular to the trajectory during the entire period in which the orbit is still practically circular. This case was discussed by Tsien [1]; as an example, we included an outline of the discussion in an Appendix of the present Memo.

3. The case of a thrust, constant with respect to the vertical

If R and C are constants, system (4) can be formally simplified by introducing dimensionless normalizing variables. We shall measure r in terms of r_o by writing $r = r_o x$, the accelerations in terms of g_o by writing $R = \mu g$, $C = \gamma g_o$, and the time by writing it in the form $t = \tau / n_o$, where $n_o = \sqrt{g_o/r_o}$ is the angular velocity on a circular orbit at $r = r_o$. The system of equations assumes the form

$$\frac{d^2 x}{d\tau^2} = \mu + x\left(\frac{d\varphi}{d\tau}\right)^2 - \frac{1}{x^2}$$
$$\frac{d}{d\tau}\left[x^2 \frac{d\varphi}{d\tau}\right] = x\gamma \qquad (4')$$

By setting $x\left(\frac{d\varphi}{d\tau}\right) \equiv \frac{d^2x}{d\tau^2} - \mu + \frac{1}{x^2} = Y$, we obtain $x^2 \frac{d\varphi}{d\tau} = \sqrt{x^3 Y}$, and the second equation assumes the form

$$\frac{1}{x}\,\frac{d}{d\tau}\sqrt{x^3 Y} = \gamma \qquad (6)$$

Since $x^3 Y = x^3 \frac{d^2x}{d\tau^2} - \mu x^3 + x$, the equation contains only x and its derivatives, /5 but this is a nonlinear equation, difficult to solve. The only simple case is $\gamma = 0$, in which case equation (6) assumes the form

$$x^3 \frac{d^2x}{d\tau^2} - \mu x^3 + x = \text{const}$$

and the constant reduces to unity if $\frac{d^2x}{d\tau^2} = \mu$ at the outset ($x = 1$). This is the case of a purely radial thrust.

In any other case (in which R and C, i.e., μ and γ, are constant) equation (6) cannot be reduced to a form directly amenable to a rigorous manual calculation. There exist, of course, valid approximations if the constants are very small or very large.

4. The case of a constant purely tangential thrust

This is the most correct case from an energy viewpoint. In this case system (5) assumes the form

$$\frac{dv}{dt} = T - \frac{\mu_0}{r^2} \sin\alpha$$
$$\frac{v^2}{\rho} = \frac{\mu_0}{r^2} \cos\alpha \tag{7}$$

It is convenient to introduce as the new variable the trajectory arc s (referred to the line $\varphi = 0$), in terms of which we have

$$\sin\alpha = \frac{dr}{ds}, \qquad \cos\alpha = \frac{r\,d\varphi}{ds}.$$

Since $\frac{dv}{dt} = v\frac{dv}{ds}$, the first equation becomes trivial

$$\frac{d}{ds}\left[\frac{v^2}{2} - \frac{\mu_0}{r}\right] = T \tag{8}$$

i.e.,

$$\frac{d}{dt}\left[\frac{v^2}{2} - \frac{\mu_0}{r}\right] = Tv$$

This is the energy equation.

If the angle α varies by $d\alpha$ at the two ends of an arc ds, corresponding to an azimuth variation $d\varphi$, the rotation of the tangent about a fixed reference will be $d\varphi - d\alpha$ and the radius of curvature will be expressed as /6

$$\frac{1}{\rho} = \frac{d\varphi - d\alpha}{ds} = \frac{d\varphi}{ds} - \frac{1}{\cos\alpha}\frac{d^2 r}{ds^2} = \frac{ds}{d\varphi}\left\{\left(\frac{d\varphi}{ds}\right)^2 - \frac{1}{r}\frac{d^2 r}{ds^2}\right\}$$

Since $\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\varphi}{ds}\right)^2 = 1$, we can write

$$\frac{1}{\rho} = \frac{ds}{d\varphi}\left\{\frac{1}{r^2} - \frac{1}{r^2}\left(\frac{dr}{ds}\right)^2 - \frac{1}{r}\frac{d^2 r}{ds^2}\right\}$$

The second equation (7) yields

$$v^2 = \frac{\mu_0}{r^2}\rho\cos\alpha = \frac{\mu_0}{r}\rho\frac{d\varphi}{ds}$$

hence

$$v^2 = \frac{\mu_0}{r}\,\frac{\left(\frac{d\varphi}{ds}\right)^2}{\frac{1}{r^2} - \frac{1}{r^2}\left(\frac{dr}{ds}\right)^2 - \frac{1}{r}\frac{d^2 r}{ds^2}} = \mu_0\,\frac{1 - \left(\frac{dr}{ds}\right)^2}{r\left\{1 - \left(\frac{dr}{ds}\right)^2 - r\frac{d^2 r}{ds^2}\right\}}$$

Equation (8) yields

$$v^2 = 2Ts + \frac{2\mu_0}{r} + C$$

If the acceleration is imparted (at s = 0) to a vehicle describing a circular orbit of radius r_o, where $v^2 = \frac{\mu_o}{r_o}$, the constant will be equal to $-\frac{\mu_o}{r_o}$ and we obtain

$$2Ts + \frac{2\mu_0}{r} - \frac{\mu_0}{r_0} = \mu_0 \frac{1 - \left(\frac{dr}{ds}\right)^2}{r\left\{1 - \left(\frac{dr}{ds}\right)^2 - r\frac{d^2 r}{ds^2}\right\}}$$

This is a nonlinear equation containing r and s only.

It was discussed by Benney [2].

It is possible to obtain more amenable equations for a numerical calculation by proceeding from system (4') for the case that the acceleration is split into its radial and circumferential components: /7

$$\ddot{x} = \mu + x\dot{\varphi}^2 - \frac{1}{x^2}$$

$$\frac{d}{d\tau}(x^2\dot{\varphi}) = x\gamma \qquad (4')$$

where the dots above the symbols denote the derivatives with respect to the dimensionless variable $\tau = \sqrt{g_o/r_o}\, t$. If the acceleration is tangential, we simply have $\mu = \lambda \sin\alpha$, $\gamma = \lambda\cos\alpha$, where λ is the absolute value of the (normalized) acceleration and α is the angle between the tangent and the horizontal, i.e.,

$$\frac{dr}{dt} = v \sin\alpha \quad , \; r\frac{d\varphi}{dt} = v\cos\alpha$$

Denoting by ψ the normalized velocity $\psi = v/\sqrt{g_o r_o}$ (ratio of the absolute velocity to the circular velocity on the original orbit), the above formulas go over into

$$\dot{x} = \psi \sin\alpha \; , \; x\dot{\varphi} = \psi\cos\alpha$$

and, by substituting into (4'), where the second equation is written as

$$(\dot{x}\dot{\varphi} + x\ddot{\varphi}) \equiv \frac{d}{d\tau}(x\dot{\varphi}) = \gamma - \dot{x}\dot{\varphi}$$

we find

$$\frac{d}{d\tau}(\psi\sin\alpha) \equiv \dot{\psi}\sin\alpha + \psi\dot{\alpha}\cos\alpha = \mu + \frac{\psi^2\cos^2\alpha}{x} - \frac{1}{x}$$

$$\frac{d}{d\tau}(\psi\cos\alpha) \equiv \dot{\psi}\cos\alpha - \psi\dot{\alpha}\sin\alpha = \gamma - \frac{\psi^2\sin\alpha\cos\alpha}{x}$$

and finally, by solving with respect to $\dot{\psi}$ and $\dot{\alpha}$,

$$\left.\begin{aligned}
\dot{\psi} &= \lambda - \frac{\sin\alpha}{x^2} \\
\dot{\alpha} &= \left(\frac{\psi}{x} - \frac{1}{\psi x^2}\right)\cos\alpha \\
\dot{x} &= \psi \sin\alpha \\
\dot{\varphi} &= \frac{\psi\cos\alpha}{x}
\end{aligned}\right\} \qquad (9)$$ /8

This is a system of 4 first-order equations, fully amenable to a computer calculation. The system is completely general, i.e., λ (which denotes $F/(mg_o)$, where F is the thrust) can be either constant or variable.

5. Formulation in terms of orbital parameters

If the thrust is in the plane, all the osculating orbits have the same inclination and the same ascending node; hence the inclination i and the right ascension Ω of the node remain constant. The four remaining orbital parameters, i.e., a, e, ω(semi major axis, eccentricity, perigee anomaly with respect to the node) and $\chi = -nt_o$ (where t_o is the perigee transit time and n the mean angular velocity on the osculating orbit) are variable.

The orbit is fully specified by a (t), e(t) and ω (t); in fact, each group of three values a, e and ω specifies an ellipse of size a, shape e, and orientation ω; the envelope of these ellipses is the orbit. In order to define a, e and ω as a function of t, we need, however, a fourth variable, for example the true anomaly θ or the eccentric anomaly E, and in order to express the latter quantity as a function of time we must know the integration constant χ, which is variable during the manoeuver.

The problem simplifies if the variation of a and e can be regarded as slow as compared to the variation of the anomalies, i.e., as compared to the orbital period; if this is the case, it is convenient to neglect the "fine" variation of the parameters (during the same revolution) and replace the functions of the anomalies by their mean values over an osculating orbit. In this case it is sufficient to know the equations for da/dt and de/dt, since the two equations do not contain other parameters. But, as we already noted, such a procedure involves numerous difficulties, since the validity of substituting functions of θ or E by their mean values has not been properly established a priori. /9

The variation of the orbital parameters in terms of "manoeuver" acceleration is described by first-order equations of Gauss. For deriving them it is possible to use, in addition to classical works, also the Technical Memo ELDO F 33, § 11 and 13; the numbers in square brackets in the equations below refer to the corresponding formulas of this Memo.

The equations for the parameters a and e are

$$\frac{da}{dt} = \frac{2a^2}{\sqrt{\mu_o p}} \left[Re \sin\theta + C(1+e\cos\theta) \right] \qquad [(34)] \tag{10}$$

$$\frac{de}{dt} = \sqrt{\frac{p}{\mu_o}} \left[R \sin\theta + C(\cos\theta + \cos E) \right] \qquad [(35)] \tag{11}$$

This system of two equations has three unknowns a, e, θ, and thus it is not sufficient for a solution. We can obtain a sufficient system by introducing a third unknown, for example the perigee anomaly ω, described by the equation

$$\frac{d\omega}{dt} = \frac{1}{e}\sqrt{\frac{p}{\mu_o}} \left\{ C \sin\theta \frac{2+e\cos\theta}{1+e\cos\theta} - R\cos\theta \right\} \qquad [(39)] \tag{12}$$

and by expressing the derivative of θ in terms of a, e and ω. In its turn the derivative of θ consists on the one hand of the "natural" variation that would take place on an osculating orbit even in the absence of a manoeuver acceleration and, on the other hand, of the perigee displacement with changed sign

$$\frac{d\theta}{dt} = \sqrt{\frac{\mu_o}{p^3}}\,(1+e\cos\theta)^2 - \frac{d\omega}{dt} \tag{13}$$

(If there would exist also an acceleration component P, perpendicular to the plane, it would enter in dω/dt; but we are discussing here manoeuvers in the plane).

The four equations (10)-(13) form a system with 4 unknowns which can be solved, in principle, for a, e, ω and θ; if, however, the purpose of the manoeuver acceleration is to effect an orbital transfer, the specification of θ(t) does not present any great interest and ω(t) is likewise not important. /10

6. Secular equations

The presence of θ (or E) in the two sides of the equations specifies a "fine" variation of the orbital elements during the same revolution; this fast variation is superimposed on a slow, progressive process, which is the only one of real interest in a description of transfer. The exact procedure for a long-term description of this process amounts to the construction of a complete solution of the system of four equations and to replace, in the result, the functions of θ (or E) by their mean values over each revolution (during which θ or E vary in view of 2π). Such a procedure amounts, in fact, to considering the period of revolution as "infinitely small" as compared to the transfer time constants.

A more recent approximation, very commonly used, though less hazardous, involves the replacing of the terms in θ or E by their mean values prior to the integration; thus one obtains equations that are commonly known as "secular" equations. The validity of such an approximation is not very easy to establish a priori; the only thing that one can say is that the approximation is definitely acceptable if the amplitude of the periodic fluctuations is infinitely small as compared to the mean value.

The scope of this approximation can be illustrated by elementary considerations; if the two sides of the equations do not contain unknows (which is not the case), the approximation will certainly be correct. To give an elementary example: In the case of an equation $\frac{dy}{dt} = a + b \cos t$, the integral of this equation ($at + b \sin t$) will consist of a secular part and of a periodic part which is zero on the average; the secular part is precisely the integral of $y = a$, i.e., of the equation in which the averaging operation has been performed prior to the integration. The same applies to an equation of the form $\frac{dy}{dt} = f(y)\,[a + b \cos t]$ which can be written as $\frac{d}{dt} F(y) = a + b \cos t$ with $F(y) = \int \frac{dy}{f(y)}$, or /11
to an equation $\frac{dy}{dt}\, f(y)\, P(t)$, where P is any periodic function of t.

This is no longer true in the case of a system which could yield by elimination (if possible) equations of order higher than the first. For example, the elementary system.

$$\dot{y} = z \quad , \quad \dot{z} = y(a + b \cos t)$$

yields

$$\ddot{y} = y(a + b \cos t)$$

which is a Mathieu equation. According to Floquet's theory we know that this equation admits an integral of the form $y = e^{\mu t} P(t)$ with periodic P, hence of an average form $y = P_o e^{\mu t}$; but the exponent μ is not necessarily to $\sqrt{a}$, as it would be the case when the average is taken prior to the integration.

Taking this into consideration, it is evident in any case that the "secular" equations are much simpler than the original equations; moreover, since the averaging operation leads to the vanishing of the unknown θ (or E), the two equations for da/dt and de/dt form a complete system (the other parameters do not occur in these equations). Let us then construct these secular equations and discuss the solution, which we know to be acceptable if the thrusts are very small.

The average must be taken with respect to time, over a period of revolution, or with respect to θ or E over an interval 2π, provided that the function is multiplied by a weight factor equal to the derivative of time with respect to θ (or E):

$$\frac{1}{T_o} \int_0^{T_o} F\, dt = \frac{1}{2\pi} \int_0^{2\pi} F \frac{dt}{d\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} F \frac{dt}{dE}\, dE$$

where F is a function of time via θ or E.

The "period of revolution" can be approximated by the period of the osculating orbit, /12
which is $T_o = 2\pi/n$, with $n = \sqrt{\mu_o/a^3}$. In terms of n we have

$$dt = \frac{(1-e^2)^{3/2}}{n} \frac{1}{(1+e\cos\theta)^2} d\theta = \frac{1 - e\cos E}{n} dE \quad \text{whence}$$

$$\tilde{F} = \frac{n}{2\pi} \int_{-\pi/n}^{\pi/n} F\,dt = \frac{(1-e^2)^{3/2}}{2\pi} \int_{-\pi}^{\pi} \frac{F}{(1+e\cos\theta)^2} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(1-e\cos E)dE \tag{14}$$

(We preferred to "center" the integration interval at the "zero" value of the variable; this simplifies the calculation of the integrals, which simply vanish if the integrand is odd).

For the use of (14) we recall the table of elementary relations:

$$\sin E = \sqrt{1-e^2}\,\frac{\sin\theta}{1+e\cos\theta} \quad ; \quad \sin\theta = \sqrt{1-e^2}\,\frac{\sin E}{1-e\cos E}$$

$$\cos E = \frac{e+\cos\theta}{1+e\cos\theta} \quad ; \quad \cos\theta = \frac{\cos E - e}{1 - e\cos E}$$

$$(1 - e\cos E)(1 + e\cos\theta) = 1 - e^2$$

$$\frac{dE}{d\theta} = \frac{\sqrt{1-e^2}}{1+e\cos\theta} = \frac{1-e\cos E}{\sqrt{1-e^2}}$$

7. Thrust, decomposed into R and C

The derivation of the secular equations on the basis of (10) and (11) is elementary; we shall see presently that the mean values of the coefficients of R, which are odd, are simply vanishing; this leads to the conclusion that the radial component has no effect on long-term transfer.

On the other hand we know that this is not true; if R exceeds $g_o/8$ (or $g_o = \mu_o/a^2$ /13 in the case of an original circular orbit of radius a), a transfer of orbit can be effected by a purely radial thrust. If $R < g_o/8$, the secular law of variation of the orbital parameters is periodic with a long period; hence it would be incorrect to conclude that a and e are constant.

This reasoning shows once again the weakness of the secular approximation, unless the accelerations are extremely small; for example in the case R = const a direct solution shows that the periodic variation of a is indeed very small (of the order of R/g) if R is small.

The mean values of the coefficients of C in (9) and (10) are easy to evaluate:

$$\widetilde{\cos\theta} = \frac{(1-e^2)^{3/2}}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta}{(1+e\cos\theta)^2}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos E - e)\,dE = -e$$

$$\widetilde{\cos E} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - e\cos E)\cos E\,dE = -\frac{e}{2}$$

$$\widetilde{1 + e\cos\theta} = \frac{(1-e^2)^{3/2}}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{1 + e\cos\theta} = \frac{1-e^2}{2\pi}\int_{-\pi}^{\pi}dE = 1 - e^2.$$

By substituting into (9) and (10), we obtain the secular equations

$$\frac{\widetilde{da}}{dt} = 2a\sqrt{\frac{p}{\mu_0}}\,C$$

$$\frac{\widetilde{de}}{dt} = -\frac{3}{2}\,e\sqrt{\frac{p}{\mu_0}}\,C \tag{15}$$

We could likewise obtain (if the formula were of interest to us)

$$\frac{\widetilde{d\omega}}{dt} = \sqrt{\frac{p}{\mu_0}}\,R\,.$$

These equations naturally coincide with the equations used by Burt [3].

8. Thrust, decomposed into T and N /14

The quantities R and C can be replaced by their expression (1) in terms of the tangential and normal components of the acceleration, thus yielding at once instead of (9) the formula

$$\frac{da}{dt} = \frac{2a^2}{\mu_0}\,vT \tag{16}$$

where v is the instantaneous velocity expressed by

$$v^2 = \frac{\mu_0}{p}(1+e^2+2e\cos\theta) = \frac{\mu_0}{a}\,\frac{1 + e\cos E}{1 - e\cos E}$$

formula (16) being nothing else but formula (3) expressing the fact that vT is the derivative of the energy.

Similarly, by writing (10) in the form

$$\frac{de}{dt} = \frac{1}{e}\sqrt{\frac{p}{\mu_0}}\,\left[R\,e\sin\theta + C(1 + e\cos\theta - 1 + e\cos E)\right]$$

where $1-e\cos E = r/a$, we immediately obtain (in view of (1) and (2)):

$$\frac{de}{dt} = \frac{pv}{\mu_o e} T - \frac{1}{e} \frac{r}{av} \left\{ T(1 + e \cos \theta) - Ne \sin \theta \right\}$$

Since $r(1+e\cos\theta) = p$, the coefficient of T can be simplified, i.e.,

$$\frac{p}{ev} \left\{ \frac{v^2}{\mu_o} - \frac{1}{a} \right\} = \frac{1}{ev} \left[1 + e^2 + 2e \cos \theta - 1 + e^2 \right] = \frac{2}{v} (e+\cos \theta)$$

which finally yields

$$\frac{de}{dt} = \frac{1}{v} \left\{ 2(e + \cos \theta)T + N \frac{r}{a} \sin \theta \right\}. \qquad (17)$$

Formulas (16) and (17) are the equations utilized by King-Hele [4]; the difference in the sign of the normal component is simply due to the fact that King-Hele takes the positive normal towards the interior.

The rate of displacement of the perigee ($d\omega/dt$) has apparently a complicated expression; it can be simplified, however, by a small manual effort, yielding /15

$$\frac{d\omega}{dt} = \frac{1}{ev} \left\{ 2T \sin \theta - N(e+ \cos E) \right\} \qquad (18)$$

9. Secular equations

As in the previous case, let us write the secular equations by replacing the functions of θ and E by their mean values. For substituting into (16) we must evaluate

$$\tilde{v} = \frac{1}{2\pi} \sqrt{\frac{\mu_o}{a}} \int_{-\pi}^{\pi} \sqrt{\frac{1+e \cos E}{1-e \cos E}} (1-e \cos E)dE = \frac{2}{\pi}\sqrt{\frac{\mu_o}{a}} \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \cos^2 E}\, dE = \frac{2}{\pi}\sqrt{\frac{\mu_o}{a}}\, \mathbf{E}(e).$$

where $\mathbf{E}$ is a complete elliptic integral of the second kind, with modulus e. This yields

$$\frac{d\tilde{a}}{dt} = \frac{2a^{3/2}}{\sqrt{\mu_o}} \frac{2}{\pi} \mathbf{E} \cdot T \qquad (19)$$

which reduces in the case of very small e to

$$\frac{d\tilde{a}}{dt} \simeq \frac{2a^{3/2}}{\sqrt{\mu_o}} T .$$

The coefficient of T in (17) is $2\frac{e + \cos\theta}{v} = 2(1 - e^2)\sqrt{\frac{a}{\mu_o}}\frac{\cos E}{\sqrt{1 - e^2\cos^2 E}}\, dE$; its mean value is

$$\frac{1-e^2}{\pi}\sqrt{\frac{a}{\mu_o}}\int_{-\pi}^{\pi}\frac{\cos E - e\cos^2 E}{\sqrt{1-e^2\cos^2 E}}\, dE = -\frac{4}{\pi}\frac{1-e^2}{e}\sqrt{\frac{a}{\mu_o}}\int_0^{\frac{\pi}{2}}\frac{e^2\cos^2 E}{\sqrt{1-e^2\cos^2 E}}\, dE$$

$$= -\frac{4}{\pi}\frac{1-e^2}{e}\sqrt{\frac{a}{\mu_o}}\left[\mathbf{K} - \mathbf{E}\right]$$

where **K** is a complete integral of the first kind.

The mean value of the coefficient of N is zero; hence we obtain

$$\widetilde{\frac{de}{dt}} = -\frac{4}{\pi}\frac{1-e^2}{e}\sqrt{\frac{a}{\mu_o}}\left[\mathbf{K} - \mathbf{E}\right]\cdot T \; . \tag{20}$$ /16

If we evaluate the mean value of $d\omega/dt$ (which is of little use), we shall encounter a surprise. The mean value of the coefficient of T is zero; hence we obtain

$$\widetilde{\frac{\cos E}{v}} = \frac{1}{2\pi}\sqrt{\frac{a}{\mu_o}}\int_{-\pi}^{\pi}\cos E\sqrt{\frac{1-e\cos E}{1+e\cos E}}\,(1 - e\cos E)\, dE$$

$$= \frac{1}{2\pi}\sqrt{\frac{a}{\mu_o}}\int_{-\pi}^{\pi}\frac{\cos E(1-e\cos E)^2}{\sqrt{1-e^2\cos^2 E}}\, dE$$

$$= -\frac{4}{\pi}\sqrt{\frac{a}{\mu_o}}\frac{1}{e}\int_0^{\pi/2}\frac{e^2\cos^2 E}{\sqrt{1-e^2\cos^2 E}}\, dE = -\frac{4}{\pi}\sqrt{\frac{a}{\mu_o}}\frac{1}{e}\left[\mathbf{K} - \mathbf{E}\right].$$

On the other hand,

$$\widetilde{\frac{1}{v}} = \frac{1}{2\pi}\sqrt{\frac{a}{\mu_o}}\int_{-\pi}^{\pi}\sqrt{\frac{1-e\cos E}{1+e\cos E}}\,(1-e\cos E)\, dE = \frac{1}{2\pi}\sqrt{\frac{a}{\mu_o}}\int_{-\pi}^{\pi}\frac{(1-e\cos E)^2}{\sqrt{1-e^2\cos^2 E}}\, dE$$

$$= \frac{2}{\pi}\sqrt{\frac{a}{\mu_o}}\int_0^{\frac{\pi}{2}}\frac{1+e^2\cos^2 E}{\sqrt{1-e^2\cos^2 E}}\, dE = \frac{2}{\pi}\sqrt{\frac{a}{\mu_o}}\left[2\mathbf{K} - \mathbf{E}\right].$$

Hence we obtain

$$\widetilde{\frac{d\omega}{dt}} = -\frac{2}{\pi}\sqrt{\frac{a}{\mu_o}}\left\{2\mathbf{K} - \mathbf{E} - 2\frac{\mathbf{K} - \mathbf{E}}{e^2}\right\} N$$

If e is small, this yields

$$\widetilde{\frac{d\omega}{dt}} = -\sqrt{\frac{a}{\mu_o}}\left\{\frac{3e^2}{8} + \frac{3e^4}{32} + \ldots\right\} N$$

which is not compatible with the conclusion found in the case of a radial thrust, i.e.,

$$\frac{\widetilde{d\omega}}{dt} = \sqrt{\frac{p}{\mu_0}}\, R . \tag{22}$$

If the eccentricity is very small, N and R tend to coincide, but (21) yields for $\frac{d\omega}{dt}$ a value tending to zero, while (22) yields a finite rate of variation. This is due to the following reason: A constant acceleration N does not yield only a radial component $N\cos\alpha = N\frac{\sqrt{1-e^2}}{\sqrt{1-e^2\cos^2 E}}$, but also a circumferential component $C = -N\sin\alpha = -N\frac{e\sin E}{\sqrt{1-e^2\cos^2 E}}$; by introducing these expressions into (11), the effects of the resultant C and R tend to compensate themselves on the average, to within a term of the order of e^2 approximately. /17

This is an almost absurd conclusion, since in the case of zero e the quantity N would yield only an R = N, which shows the subtlety of a "secular average" operation. As a matter of fact, in the presence of an R or of an N, the eccentricity cannot remain zero, thus causing a discrepancy between the two accelerations; this explains, in a form not very accessible to intuition, the difference between the results.

10. Integration of secular equations

In the secular equations for the RC case the quantity R does not occur; only the circumferential component appears to be useful. The equations are

$$\begin{aligned} \frac{da}{dt} &= 2a\sqrt{\frac{p}{\mu_0}}\, C = 2a^{3/2}\sqrt{1-e^2}\,\frac{C}{\sqrt{\mu_0}} \\ \frac{de}{dt} &= -\frac{3}{2}\, e\sqrt{\frac{p}{\mu_0}}\, C = -\frac{3}{2}\, e\sqrt{1-e^2}\cdot\sqrt{a}\cdot\frac{C}{\sqrt{\mu_0}} \end{aligned} \tag{23}$$

(The "secular average" sign has been dropped for reasons of simplicity).

The system is very easy to solve; by taking the ratio of two sides, we obtain

$$\frac{de}{da} = -\frac{3}{4}\frac{e}{a} \tag{24}$$

i.e., (denoting the initial values by the subscript "0")

$$e = e_0 \left(\frac{a_0}{a}\right)^{3/4} \tag{25}$$

and by substituting into the first equation (23) /18

$$\frac{da}{dt} = 2a^{3/2}\cdot\sqrt{1 - e_0^2\left(\frac{a_0}{a}\right)^{3/2}}\;\frac{C}{\sqrt{\mu_0}} \tag{26}$$

These are the equations utilized by Burt [3]. Equation (26) can be solved by a quadrature, yielding t as a function of a. If the original orbit is circular, the integral will be elementary:

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a_o}} - \frac{C}{\sqrt{\mu_o}}\, t$$

If e_o is not zero, equation (26) can still be integrated in explicit form, though not very convenient, i.e., by taking $\sqrt{a_o/a}$ as the new variable, the time will be expressed in the form of an elliptic integral of the first kind with modulus 75°. It is not appropriate to dwell on this formal solution, whose cumbersomeness is out of proportion with the uncertainty of the approximation; since e_o is small, it is always possible to expand $\left\{1 - e_o^2\left[a_o/a\right]^{3/2}\right\}^{-1/2}$, and to integrate the series.

When a_o/a is known as a function of time, the eccentricity will be expressed by (25), which shows that the eccentricity decreases constantly when the orbit is enlarged. The eccentricity will always remain zero if e_o were zero, which is a result that carries little conviction.

The case of a purely tangential thrust can also be treated in secular form; the secular equations are

$$\frac{da}{dt} = \frac{2a^{3/2}}{\sqrt{\mu_o}} \; \frac{2}{\pi}\; \mathbf{E} \,.\, T \tag{19}$$

$$\frac{de}{dt} = -\frac{4}{\pi}\;\frac{1-e^2}{e}\sqrt{\frac{a}{\mu_o}}\,\left[\mathbf{K} - \mathbf{E}\right] .\, T \tag{20}$$

By taking the ratio of two sides of the equation, we immediately obtain

$$\frac{da}{a} = -\frac{e}{1-e^2}\;\frac{\mathbf{E}}{\mathbf{K}-\mathbf{E}}\; de \tag{27}$$

Despite its prohibitive look, this equation can be integrated "at sight", $eE/(1-e^2)$ being in fact nothing else but the derivative of **K-E** with respect to e. /19

Hence

$$\mathbf{K} - \mathbf{E} = \frac{C}{a} \tag{28}$$

K-E is an increasing function of e; the eccentricity decreases, even in the present case, when the orbit is enlarged; but at $e_o = 0$ the constant C must be zero, and the eccentricity will always remain zero!

If the initial eccentricity is not zero, we have $C = a_o[\mathbf{K}-\mathbf{E}]_{e=e_o}$; by substituting into (20) the value of a from (28), we obtain the equation

$$\frac{de}{dt} = -\frac{4}{\pi}\,\frac{1-e^2}{e}\sqrt{\frac{C}{\mu_o}}\,\sqrt{K-E}\;.T.dt \tag{29}$$

In principle, equation (29) solves the problem; by writing this equation in the form

$$\frac{e\,de}{(1-e^2)\sqrt{K-E}} = -\frac{4}{\pi}\sqrt{\frac{C}{\mu_o}}\cdot T\,dt \tag{30}$$

it could yield the time by a direct quadrature, as a function of e.

The quardrature for the left-hand side of (30) can be performed (numerically, of course) <u>once and for all</u>, since this term does not contain other parameters. After the equation has been solved with respect to e, hence with respect to K-E, we can obtain <u>a</u> directly from (28).

These equations are basically the same as those obtained by King-Helle [4], who does not use, however, the symbols of elliptic integrals, but replaces them by corresponding series, generally truncated at the second-order term. In particular, he solves "once and for all" an approximated equation (26), obtaining $\frac{a}{a_o}$ as a function of $\frac{e}{e_o}$ [4, Fig. 2]; he concludes that the shape of the curve does not appreciably change when e_o varies. (But the solution will be erroneous if $e_o = 0$!).

It can be noted (King-Hele) that by writing the equation which expresses de/da (the variation of the eccentricity as a function of the major axis) in the form (24) or (27), <u>the acceleration of manoeuver is eliminated</u>; the law e(a) remains the same when the thrust is changed from one revolution to another (we are dealing with equations that are valid /20 on the average) or perhaps switched off at intervals.

11. Some conclusions

We are drawn to the conclusion that the "planetary" formulation, which studies the behavior of the parameters in time, is amenable to a fairly elegant mathematical treatment (if we proceed from the secular equations), though it is a bit dangerous to accept the results without any reservations.

On the other hand we deem it useful (in any case) to check the results of the "planetary" formulation by a simultaneous integration of the "direct" equations with the same parameters; this can be done independently of the solution of the actual problem, by selecting the appropriate parameters, in order to be able to establish the limits of validity. (We could see, for example, that the planetary formulation yields a zero secular variation of a and e as a result of a purely radial acceleration, which is not true at all and should be further examined).

II. CORRECTIONS OUTSIDE THE PLANE

12. Direct formulation

/21

If the acceleration of manouever has a component along the normal to the osculating plane (a "binormal" component B), the orbit will change its plane, i.e., the planetary coordinates i (inclination) and Ω (right ascension of the node) will be variable. In direct notation we can write the differential equations in terms of the radius vector r, the azimuth φ, and the latitude β, implying for example the components R, C and B. In order to formulate the equations correctly, it is convenient to introduce a provisional Cartesian coordinate system

$$x = r\cos\beta\cos\varphi$$

$$y = r\cos\beta\sin\varphi$$

$$z = r\sin\beta$$

with x and y in the original osculating plane, the x-coordinate passing through the vehicle. Hence the initial values of the original coordinates will be $x = r$, $y = z = 0$, $\beta = 0$ and $\varphi = 0$.

Thus we obtain at once

$$\ddot{x} = \ddot{r} - r\dot{\beta}^2 - r\dot{\varphi}^2$$

$$\ddot{y} = 2\dot{r}\dot{\varphi} + r\ddot{\varphi}$$

$$\ddot{z} = 2\dot{r}\dot{\beta} + r\ddot{\beta}$$

and the Cartesian equations

$$\ddot{x} = -\mu_0 x/r^3 + R,\quad \ddot{y} = -\mu_0 y/r^3 + C,\quad \ddot{z} = -\mu_0 z/r^3 + B$$

go over into

$$\ddot{r} = R - \frac{\mu_0}{r^2} + r\dot{\varphi}^2 + r\dot{\beta}^2$$

$$\frac{d}{dt}(r^2\dot{\varphi}) = r\,C \qquad (31)$$

$$\frac{d}{dt}(r^2\dot{\beta}) = r\,B$$

These equations are very complicated, even on the assumption that the acceleration components are constant; it is not even possible to reduce the system to a single equation containing r, as we did in the case $B = 0$. /22

The system can be solved, of course, by a numerical calculation, but this would not facilitate the obtaining of synthetic conclusions.

13. Formulation in planetary coordinates

On the other hand a formulation in planetary coordinates appears to be very simple; the B component does not occur in the expressions for $\frac{da}{dt}$ and $\frac{de}{dt}$, so that the conclusions obtained hitherto with regard to the major axis and the eccentricity remain unchanged; a correction term appears in the expression specifying $\frac{d\omega}{dt}$, which goes over into

$$\frac{d\omega}{dt} = \frac{1}{e}\sqrt{\frac{p}{\mu_0}}\left\{ C \sin\theta \, \frac{2+e\cos\theta}{1+e\cos\theta} - R\cos\theta \right\} - \sqrt{\frac{p}{\mu_0}} \cot i . \frac{\sin(\omega+\theta)}{1+e\cos\theta} B$$

(A slightly manipulated equation (39) of TM 33).

While the determination of $\omega(t)$ is not of great interest in the case of plane corrections, ω is needed for the determination of orbital-plane variations under the effect of the acceleration B. The two planetary coordinates i and Ω are specified, in fact, by the equations

$$\frac{di}{dt} = \frac{r}{\sqrt{\mu_0 p}} \cos(\omega+\theta) B = \sqrt{\frac{p}{\mu_0}} \frac{\cos(\omega+\theta)}{1+e\cos\theta} B \tag{32}$$

$$\frac{d\Omega}{dt} = \frac{r}{\sqrt{\mu_0 p}} \frac{\sin(\omega+\theta)}{\sin i} B = \sqrt{\frac{p}{\mu_0}} \frac{1}{\sin i} \frac{\sin(\omega+\theta)}{1+e\cos\theta} B \tag{33}$$

These are the equations [(37)] and [(38)] of TM F 33. /23

If the acceleration B acts alone, the secular mean of $d\omega/dt$ will be

$$\widetilde{\frac{d\omega}{dt}} = -\sqrt{\frac{p}{\mu_0}} \cot i \frac{B}{2\pi} \int_{-\pi}^{\pi} \frac{\sin\omega(\cos E - e) + \cos\omega \sqrt{1-e^2}\sin E}{1-e^2} (1-e\cos E)dE$$

$$= \frac{3}{2} e \frac{a}{\sqrt{\mu_0 p}} \cot i \sin\omega . B$$

and, similarly, the mean of $\widetilde{d\Omega/dt}$

$$\widetilde{\frac{d\Omega}{dt}} = -\frac{3}{2} e \frac{a}{\sqrt{\mu_0 p}} \frac{\sin\omega}{\sin i} B \tag{34}$$

equation (34) corresponds to equation (27) of Burt [3].

Analogously,

$$\widetilde{\frac{di}{dt}} = \sqrt{\frac{p}{\mu_o}}\ \frac{B}{2\pi} \int_{-\pi}^{\pi} \frac{\cos\omega(\cos E - e) - \sin\omega\sqrt{1-e^2}\sin E}{1 - e^2} (1 - e\cos E)\, dE$$

$$= -\frac{3}{2} e \frac{a}{\sqrt{\mu_o p}} \cos\omega \cdot B \qquad (35)$$

corresponds to (24) in [3].

For a circular orbit, all these orbital-plane changes are zero on the average; in order to avert this, Burt proposes a reversal of the direction of the acceleration B at the points at which $\frac{di}{dt}$ changes sign, i.e., according to (32) at the points where cos ($\omega + \theta$) vanishes. By similar considerations, Burt suggests sign reversals of certain variables in the case that the (instantaneous, not secular) derivative of the variable of interest vanishes. This evidently makes it possible to prevent the secular derivative from decreasing to zero or to an excessively small value as a result of a sign reversal of the instantaneous derivative. This is evidently an efficient procedure, though of doubtful practicability. Naturally, by reversing the sign of a thrust in order to speed up the variation of one of the planetary coordinates, we may cause a reduction (or the vanishing) of the mean variation of another coordinate; this may be unimportant if only the first coordinate is of interest.

APPENDIX

/24

ORBITAL ENLARGEMENT BY A CONSTANT VERTICAL THRUST

1. Statement of problem

It is required, by proceeding from a circular orbit of geocentric radius r_o, to enlarge the orbit until (possibly) escape, by using small thrusts that are constant (during a limited time, of course). In a spherical potential field and in free space the analytic problem is very simple. The following simplifying assumptions are adopted, schematically represented in the form of 3 cases:

a) The thrust is purely radial ("vertical") and constant.
b) The thrust is purely "circumferential" (horizontal") and constant.
c) The thrust is tangential and the trajectory is constant.

Here we shall consider the case of a vertical thrust. Denoting by θ the geocentric azimuth with respect to the original vertical, the initial conditions will be

$$\left(\frac{dr}{dt}\right)_0 = 0 \quad , \quad r_0\left(\frac{d\theta}{dt}\right)_0^2 = g_0 \qquad (1)$$

where g_o is the gravitational force at the altitude of the circular orbit.

Denoting by R the radial acceleration (positive in the upward direction) and by C the circumferential acceleration (in the direction of increasing θ), we obtain the equations of motion

$$\frac{d^2r}{dt^2} = R + r\left(\frac{d\theta}{dt}\right)^2 - g_0\frac{r_0^2}{r^2}$$
$$\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right] = rC \qquad (2)$$

with the initial conditions.

2. General equation of radial (vertical) thrust

With C = 0, the second equation (2) reduces to

$$r^2\frac{d\theta}{dt} = \text{const.}$$

From the second condition (1) we obtain for the constant the value $\sqrt{g_o r_o^3}$. /25

Hence the equation for r assumes the form

$$\frac{d^2r}{dt^2} = R + g_0\frac{r_0^3}{r^3} - g_0\frac{r_0^2}{r^2} \qquad (3)$$

i.e.,

$$\frac{d}{dr}\left(\frac{dr}{dt}\right)^2 = 2R + 2g_o\frac{r_o^3}{r^3} - 2g_o\frac{r_o^2}{r^2},$$

This equation can be integrated at sight, yielding

$$\left(\frac{dr}{dt}\right)^2 = 2Rr - \frac{g_o r_o^3}{r^2} + \frac{2g_o r_o^2}{r} + C$$

The condition that dr/dt vanishes at r_o, specifies the constant

$$\left(\frac{dr}{dt}\right)^2 = 2R(r-r_o) - \frac{g_o r_o^3}{r^2} + \frac{2g_o r_o^2}{r} - g_o r_o$$

By setting $r = r_o \rho$, this equation goes over into

$$r_o\left(\frac{d\rho}{dt}\right)^2 = 2R(\rho-1) - g_o\left[\frac{1}{\rho^2} - \frac{2}{\rho} + 1\right],$$

hence

$$\frac{d\rho}{dt} = \sqrt{\frac{g_o}{r_o}}\,\frac{\sqrt{(\rho-1)(2\mu\rho^2-\rho+1)}}{\rho} \tag{4}$$

where μ denotes the ratio R/g_o.

Hence the radial velocity vanishes first of all at $\rho = 1$, which represents an initial condition, and at the two roots of

$$2\mu\rho^2 - \rho + 1 = 0 \tag{5}$$

which are complex if $8\mu > 1$. With a thrust R exceeding 1/8 of the force of gravity, it is possible to achieve any distance (until escape).

3. Thrusts smaller than $g_o/8$

If the thrust is smaller than this value ($\mu < 1/8$), the radial velocity vanishes at the first root of (5), i.e., at

$$\rho = \frac{1-\sqrt{1-8\mu}}{4\mu}. \tag{6}$$

This has the value of two, i.e., the initial (geocentric) radius can be more than doubled at the limiting value $\mu = 1/8$; at any other value of μ the maximum distance that can be reached is specified by (6). /26

The time needed to reach the radius $r_o \rho$ is specified by (4)

$$t = \sqrt{\frac{r_o}{g_o}} \int_1^{\rho} \frac{\rho\, d\rho}{\sqrt{(\rho-1)(2\mu\rho^2-\rho+1)}} = \sqrt{\frac{r_o}{2R}} \int_1^{\rho} \frac{\rho\, d\rho}{\sqrt{(\rho-1)(\rho-\rho_1)(\rho-\rho_2)}}$$

by denoting with ρ_1 and ρ_2 the two roots of (5)

$$\rho_{1,2} = \frac{1 \mp \sqrt{1-8\mu}}{4\mu}.$$

By setting $\rho - 1 = x$, the integral goes over into

$$t = \sqrt{\frac{r_o}{2R}} \int_0^{x} \frac{(x+1)\, dx}{\sqrt{x(x-x_1)(x-x_2)}}$$

with

$$x_{1,2} = \frac{1-4\mu \mp \sqrt{1-8\mu}}{4\mu}.$$

When $x<x_1$, the integral will be

$$t = \sqrt{\frac{r_o}{2R}} \left\{ \frac{2(1+x_2)}{\sqrt{x_2}} F(\varphi) - 2\sqrt{x_2}\, E(\varphi) \right\} \tag{7}$$

with $x = x_1 \sin^2\varphi$, the modulus of the elliptic integrals being expressed by $k^2 = x_1/x_2$, or, quite simply, by $k = x_1$ (since $x_1 x_2 = 1$).

The time for a complete transfer to the new orbit is

$$T = \sqrt{\frac{r_o}{2R}} \left\{ \frac{2(1+x_2)}{\sqrt{x_2}} \mathbf{K}(x_1) - 2\sqrt{x_2}\, \mathbf{E}(x_1) \right\} \tag{8}$$

where **K** and **E** denote complete elliptic integrals; the time becomes infinite (logarithmic) at $x_1 = 1$, i.e., at $8\mu = 1$. /27

The tangential velocity reached is specified by

$$r \frac{d\theta}{dt} = \frac{\sqrt{g_o r_o^3}}{r} = \frac{\sqrt{g_o r_o}}{\rho_1}$$

whereas the original velocity was $\sqrt{g_o r_o}$; the equilibrium velocity on the new orbit must be

$$\left[r\left(\frac{d\theta}{dt}\right)^2\right]_{eq} = g_o \frac{r_o^2}{r^2}$$

i.e.,

$$\left[r\,\frac{d\theta}{dt}\right]_{eq} = \frac{\sqrt{g_o r_o}}{\sqrt{\rho_1}}.$$

Hence we have a tangential velocity deficit of

$$\sqrt{g_o r_o}\left\{\frac{1}{\sqrt{\rho_1}} - \frac{1}{\rho_1}\right\}$$

i.e., a centrifugal acceleration deficit of

$$g_o\left[\frac{r_o^2}{r^2} - \frac{r_o^3}{r^3}\right] = g_o\left[\frac{1}{\rho_1^2} - \frac{1}{\rho_1^3}\right]$$

Since $2\mu\rho_1^2 - \rho + 1 = 0$, this is equal to

$$\frac{2\mu g_o}{\rho_1} = \frac{2R}{\rho_1} > R$$

since $\rho_1 < 2$. Thus the deficit is not compensated by the thrust R (except in the case that $8\mu = 1$; but in this case the time needed to attain the equilibrium radius will be infinite). Thus ρ_1 specifies in fact an apogee, after which the body begins to fall despite the presence of the thrust.

The fall trajectory naturally follows equation (3) as well, and hence equation (4), where the square root must now be taken with negative sign. Thus the fall is halted for a second time, at $\rho = 1$, i.e., at the original orbit; then the body begins to rise again. The period will be double the time T specified by (8), i.e., infinite if $8\mu = 1$. /28

At this value of the thrust the body is actually following (asymptotically) a circular orbit, at the expense, of course, of the continuous thrust compensating the deficit in tangential velocity.

Thus we arrive at the conclusion that radial thrusts above $g_o/8$ are not practicable; they can be utilized, however, for transferring the body to the apogee ρ_1, where the orbit can be stabilized by means of a tangential velocity increment

$$\Delta V = \sqrt{g_o r_o}\left\{\frac{1}{\sqrt{\rho_1}} - \frac{1}{\rho_1}\right\} \qquad (8)\ [sic]$$

Thus we would realize a transfer of circular orbit (limited, of course, to a radius smaller than double the initial radius; yet the new orbit could perhaps be used as a new parking orbit, even if the procedure is complicated).

It is of interest to evaluate the characteristic velocity corresponding to transfer from $\rho = 1$ to $\rho = \rho_1$ by means of a radial thrust μg_0 during a time T plus the increment ΔV specified by (9). Denoting by c the ejection velocity, the thrust R will be specified by

$$MR = -c\, dM/dt$$

where M is the instantaneous mass. Since R is assumed constant, we have during a time T the formula

$$c \ln (M_0/M_1) = RT$$

this is precisely the characteristic velocity with respect to the radial thrust. The characteristic velocity with respect to transfer will hence be

$$\frac{\sqrt{Rr_0}}{\sqrt{2}} \left\{ \frac{2(1+x_2)}{\sqrt{x_2}} \mathbf{K}(x_1) - 2\sqrt{x_2}\, \mathbf{E}(x_1) \right\} + \sqrt{g_0 r_0} \left\{ \frac{1}{\sqrt{1+x_1}} - \frac{1}{1+x_1} \right\}$$

In general, we may conclude that an orbital correction by means of a small radial thrust is of interest only in the case of very small corrections obtained by low values of μ. When μ is small, we have

$$x_1 = 2\mu + 8\mu^2 + 40\mu^3$$

which permits a relative increase of the geocentric radius in view of $2\mu + \ldots$. With /29
$x_2 = \frac{1}{x_1} = \frac{1}{2\mu} - 2 + \ldots$, we furthermore obtain

$$2\left(\sqrt{x_1} + \frac{1}{\sqrt{x_1}}\right)\mathbf{K} - \frac{2}{\sqrt{x_1}}\mathbf{E} \simeq \pi\sqrt{2\mu}\left\{1 + \mu + \ldots\right\}$$

and since

$$\frac{1}{\sqrt{1+x_1}} - \frac{1}{1+x_1} \simeq \mu - \frac{5}{2}\mu^2 \ldots$$

the total characteristic velocity will be expressed as

$$\sqrt{g_0 r_0}\left\{(\pi+1)\mu + \left(\pi - \frac{5}{2}\right)\mu^2, \ldots\right\}$$

in fact proportional to μ, hence to the relative increase of the orbital radius. For sufficiently low orbits, the quantity $\sqrt{g_0 r_0}$ (which represents the circular velocity) is of the order of 8000 m/sec.

The transfer time, specified by (8), goes over into

$$T = \sqrt{\frac{r_o}{2\mu g_o}}\ \pi\sqrt{2\mu}\ \ldots \simeq \pi\sqrt{\frac{r_o}{g_o}}\ \ldots$$

independent (to the first order) of the thrust, and hence of the correction value. Since $\sqrt{g_o/r_o} = \omega_o$ is the angular velocity on the original orbit, the quantity $\pi\sqrt{r_o/g_o}$ will be only one-half the initial orbital period.

4. Thrusts larger than $g_o/8$

The equation expressing the time as a function of the relative radius is the same as before

$$t = \sqrt{\frac{r_o}{g_o}} \int_1^{\rho} \frac{\rho\, d\rho}{\sqrt{(\rho-1)(2\mu\rho^2-\rho+1)}}$$

but here the trinomial in the denominator has no real zeros. The position $\rho = 2/(1+\cos\varphi)$ assigns to $\rho = 1,\ 2,\ \infty$ the angles $\varphi = 0,\ \frac{\pi}{2}$ and π; the integral goes over into

$$t = 2\sqrt{\frac{r_o}{2\mu g_o}} \int_0^{\varphi} \frac{d\varphi}{(1+\cos\varphi)\sqrt{1-k^2\sin^2\varphi}}$$

with $k^2 = 1/8\mu$. The factor in front of the integral is $\sqrt{2r_o/R}$. /30

$$t = \sqrt{\frac{2r_o}{R}} \left\{ \tan\frac{\varphi}{2}\cdot\sqrt{1-k^2\sin^2\varphi} + F - E \right\} \tag{9}$$

where F and E are elliptic integrals of the second kind, with modulus k. Formula (10) holds for all the values of ρ between 1 and ∞, but it must be used with care. For $\rho = 2$, φ assumes the value $\frac{\pi}{2}$ and the time will be expressed as

$$t_2 = \sqrt{\frac{2r_o}{R}} \left\{ \sqrt{1-k^2} + \mathbf{K} - \mathbf{E} \right\}.$$

Thus the time will be infinite (logarithmic) if $k = 1$, i.e., if $8\mu = 1$, which is a known fact.

At values higher than $\rho = 2$ (for values of μ higher than 1/8), the angle φ is larger than $\frac{\pi}{2}$; therefore it is preferable to write (10) in the form

$$t = \sqrt{\frac{2r_o}{R}} \left\{ \tan\frac{\varphi}{2}\cdot\sqrt{1-k^2\sin^2\varphi} + 2\mathbf{K} - 2\mathbf{E} - F(\pi-\varphi) + E(\pi-\varphi) \right\} \tag{10}$$

if we intend to use ordinary tables.

The vehicle reaches its escape velocity when the total energy vanishes, i.e., when $\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - g_0\frac{r_0^2}{r} = 0$; by replacing the expressions for dr/dt and dθ/dt, this formula goes over into

$$2Rr - 2Rr_0 - \frac{g_0 r_0^3}{r^2} + \frac{2g_0 r_0^2}{r} - g_0 r_0 + \frac{g_0 r_0^3}{r^2} - \frac{2g_0 r_0^2}{r} = 0$$

i.e.,

$$r^* = r_0\left(1 + \frac{g_0}{2R}\right) = r_0\left(1 + \frac{1}{2\mu}\right)$$

When $2\mu > 1$, the normalized escape radius ρ^* will be smaller than 2 and formula (10) can be utilized in its original form. Since

$$\cos\varphi^* = \frac{2-\rho^*}{\rho^*} = \frac{2\mu - 1}{2\mu + 1}$$

we find

$$\operatorname{tang}\frac{\varphi^*}{2} = \frac{1}{\sqrt{2\mu}}, \quad \sqrt{1 - k^2\sin^2\varphi^*} = \frac{2\sqrt{\mu(1+\mu)}}{2\mu+1}$$

The two roots must be taken positive. Hence

$$t^* = \sqrt{\frac{2r_0}{R}}\left\{\frac{\sqrt{2(\mu+1)}}{2\mu+1} + F(\varphi^*) - E(\varphi^*)\right\} \qquad (11)$$ /31

with $\varphi^* = \arccos\frac{2\mu-1}{2\mu+1}$

When $2\mu < 1$, formula (12) will still hold, but φ^* will lie between $\frac{\pi}{2}$ and π ; hence it might be more convenient to write

$$t^* = \sqrt{\frac{2r_0}{R}}\left\{\frac{\sqrt{2(\mu+1)}}{2\mu+1} + 2K - 2E - F(\pi - \varphi^*) + E(\pi - \varphi^*)\right\}$$

which shows that the escape time becomes infinite at $8\mu = 1$, by asymptotically stopping the enlargement at $\rho = 2$.

Formula (12) coincides with Tsien's formula ([1], formula (17)), apart from the sign of E, which is erroneously given as "+" in [1].

The characteristic velocity "employed" for escape is evidently Rt*.

5. Conclusions

The results can be summed up as follows:

With a very small radial thrust, $\mu < 1/8$, $[R < g_o/8]$, the enlargement of the orbit is limited to the relative quantity

$$\rho_1 = \frac{1 - \sqrt{1 - 8\mu}}{4\mu}, \qquad [\rho_1 = r_1/r_o]$$

which reaches the value 2 when $\mu = 1/8$. This orbital radius is reached with a tangential velocity smaller than the circular velocity; as a result, the vehicle begins to fall, after attaining the apogee ρ_1, describing a trajectory which is a mirror image of the ascending trajectory, thus rejoining the original orbit, and so on. This process occurs despite the permanence of the thrust.

If the thrust were switched off at the apogee, the vehicle would follow an elliptic orbit with a semi major axis specified by $\frac{1}{a} = \frac{2}{r_1} - \frac{v^2}{g_o r_o^2} = \frac{2}{r_1} - \frac{r_o}{r_1^2} = \frac{1}{r_o}\left[\frac{2}{\rho_1} - \frac{1}{\rho_1^2}\right]$; hence $a = r_o \frac{\rho_1^2}{2\rho_1 - 1}$; the perigee would be at $2a - r_o \rho_1 = \frac{r_o}{2\rho_1 - 1}$; when $\rho_1 > 1$, the perigee would fall below the original orbit. The constancy of the thrust permits the /32 vehicle to be maintained between the apogee and the original orbit, the trajectory being, of course, not elliptic, but rather helical.

The orbit could be made circular at r_1 by means of a tangential thrust of the apogee, in a much more efficient way than by means of a radial thrust. The tangential thrust needed in this case corresponds to the difference between the circular velocity $\sqrt{g_o r_o}/\rho_1$ and the actual velocity $\sqrt{g_o r_o}/\rho_1^2$.

At the limit $8\mu = 1$, the quantity ρ_1 is equal to 2, and a is equal to $\frac{4}{3} r_o$; if the thrust would cease at this point (which is reached in fact only after an infinite time), the perigee would drop to $\frac{2}{3} r_o$; the velocity increment ΔV, needed to make the orbit circular at $2r_o$, would be equal to $\sqrt{g_o r_o}/2$, i.e., half the original orbital velocity.

The time needed to reach the apogee is of the order of the initial half-period in the case of slow thrusts, but it becomes infinite (logarithmic) at $8\mu = 1$.

When the thrust exceeds the value $\mu = 1/8$, there are no limits on the increase in distance; in particular, at a distance

$$\rho_1 = 1 + \frac{1}{2\mu}$$

the vehicle attains its escape velocity. The time needed to attain the escape velocity is specified by formula (12), which can be rewritten as

$$t^* = \frac{T_0}{2\pi}\sqrt{\frac{2}{\mu}}\left\{\frac{\sqrt{2(\mu+1)}}{2\mu+1} + F(\varphi^*) - E(\varphi^*)\right\}$$

the modulus of the elliptic integrals being specified by $k^2 = 1/8\mu$, and φ^* being defined by $\arccos\frac{2\mu-1}{2\mu+1}$, taking values between 0 and π.

When $2\mu = 1$, φ^* is equal to $\pi/2$; escape is achieved at a radius $2r_0$ in a time

$$t^* = \frac{T_0}{\pi}\left\{\frac{\sqrt{3}}{2} + K\left(\frac{1}{2}\right) - E\left(\frac{1}{2}\right)\right\}$$

i.e., during $T_0/3$ approximately. The time is smaller if the thrust exceeds one-half the gravitation g_0.

REFERENCES

/33

1. Tsien, H.S.: "Take-Off from Satellite Orbit", ARS Journal, Vol. 23, No. 4, p. 233, July-August 1953.
2. Benney, D.J.: "Escape from a Circular Orbit Using Tangential Thrust", Jet Propulsion Journal of the ARS, Vol. 28, No. 3, p. 167, March 1958.
3. Burt, E.G.C.: "On Space Manoeuvres with Continuous Thrust", Planetary and Space Sciences, Vol. 15, p. 103, 1967.
4. King-Hele, D.G.: "The Enlargement of Elliptic Satellite Orbit by Continuous Micro-Thrust", Proc. of the XIVth International Astronautical Congress, Paris, p. 155, 1963.

Translated for the National Aeronautics and Space Administration by Scripta Technica, Inc. NASw-1694.